Determinacy, Large Cardinals, and Inner Models

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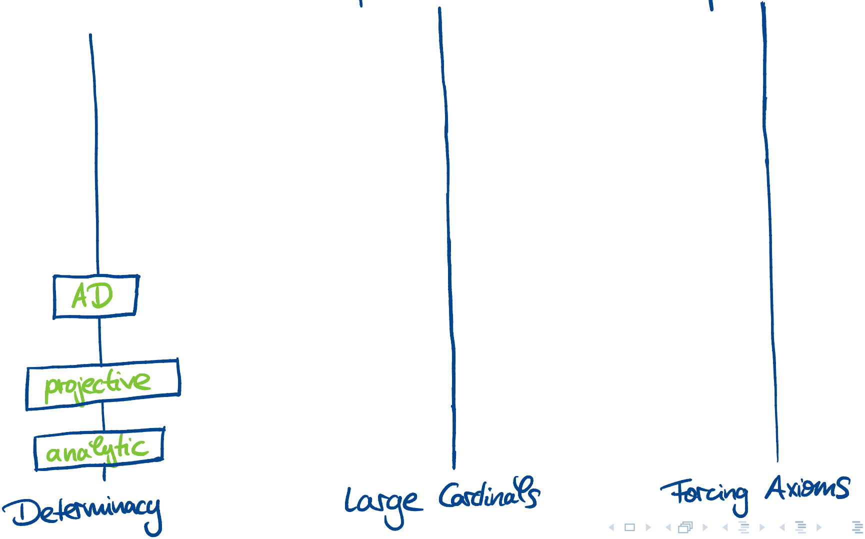
How far are these axioms from ZFC? "Steel's Program"

Consider hierarchies of these axioms and compare their strength.



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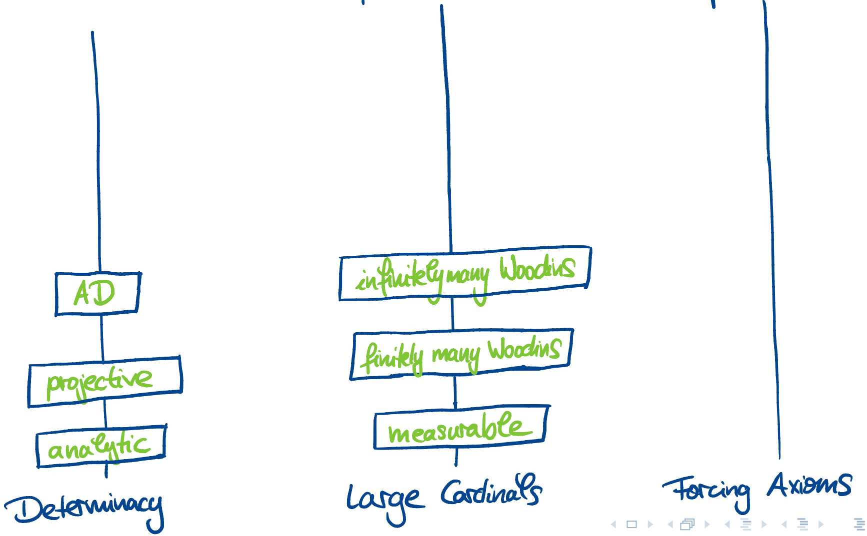

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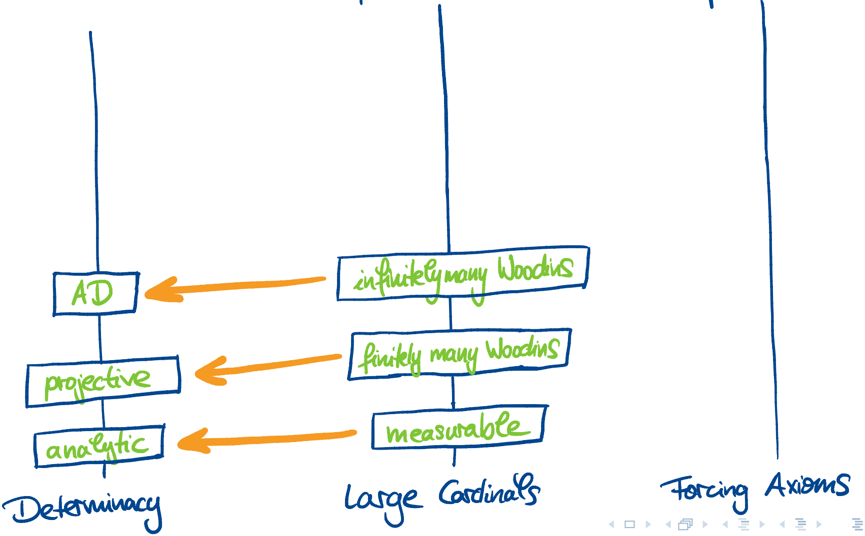

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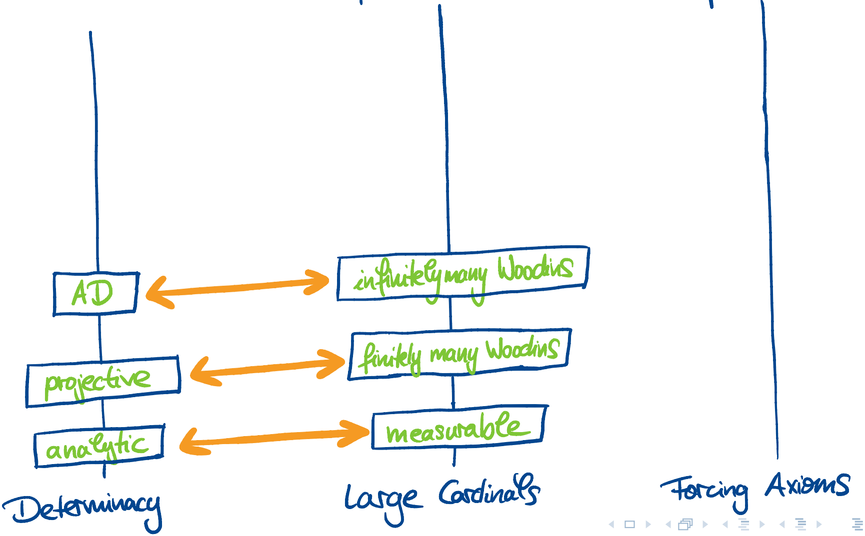

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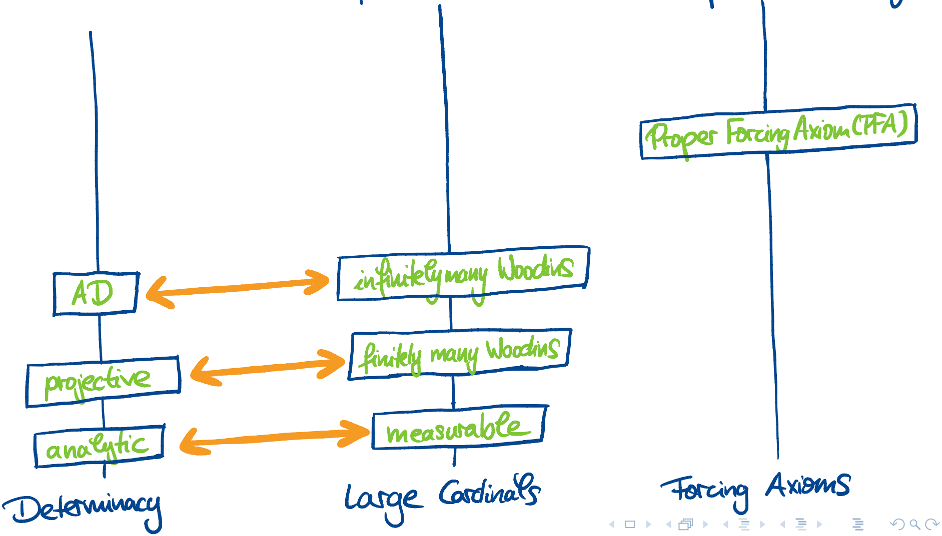

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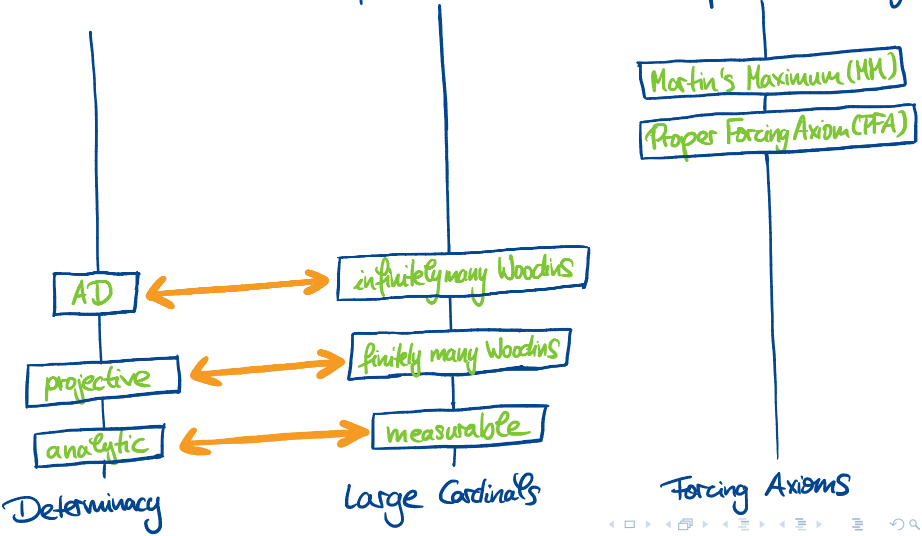

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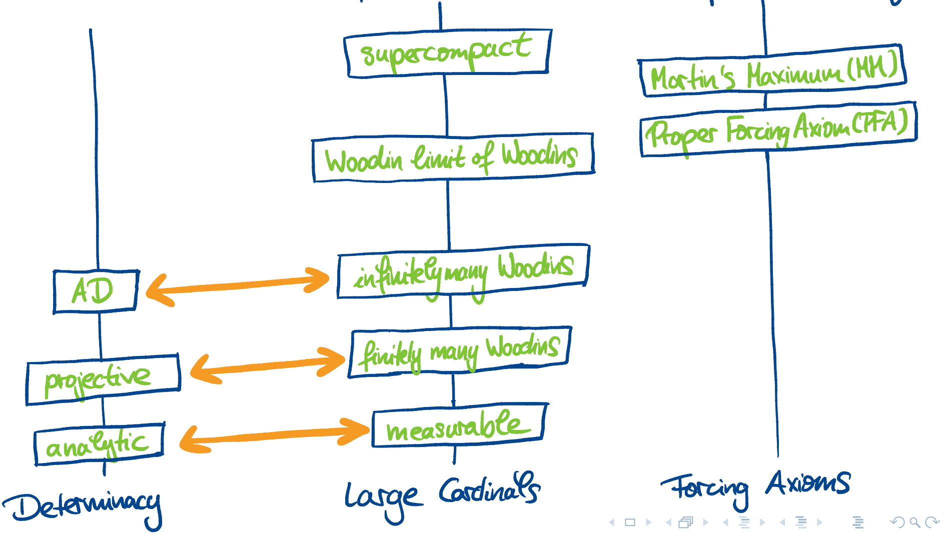

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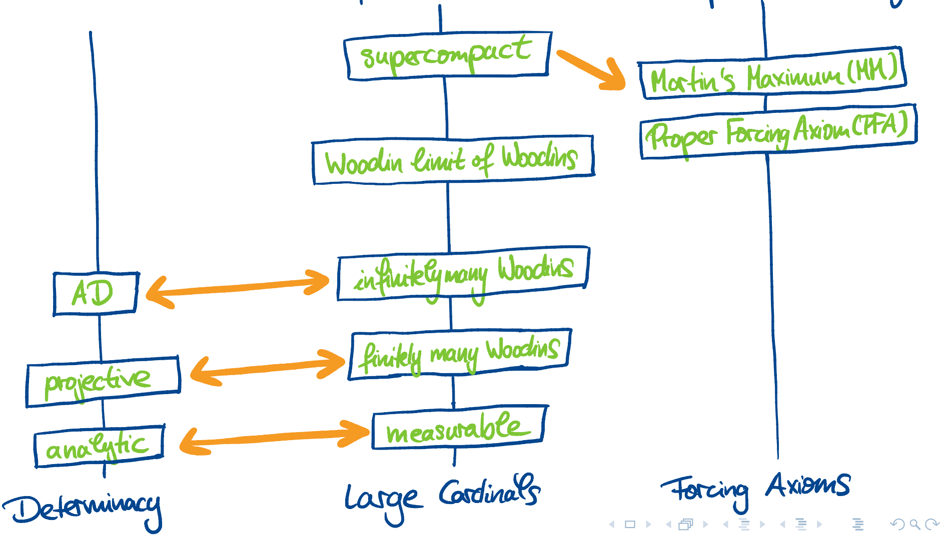

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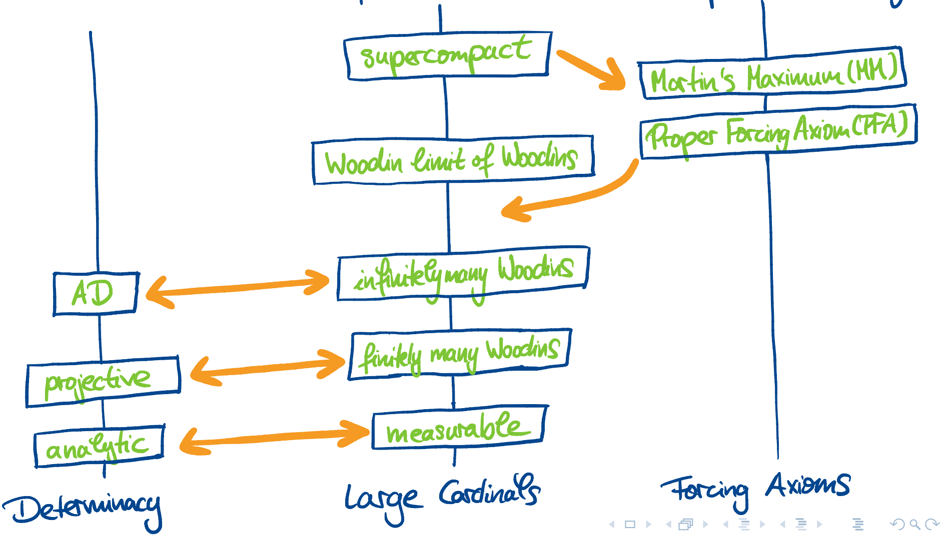

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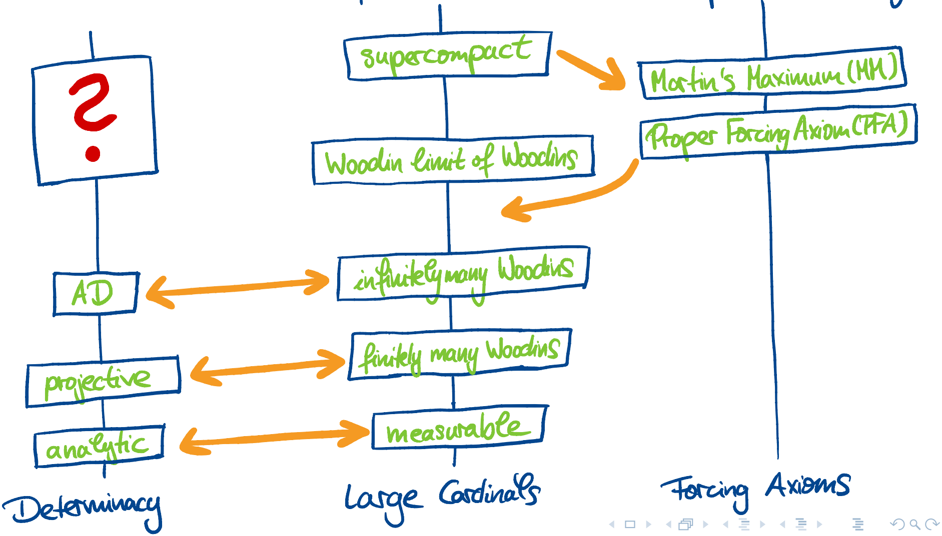

Two scenarios

What axiom(s) could fill the gap in the determinacy hierarchy?

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Long games

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Strong models

Long games

Two scenarios

Standard games:

I	$n_0 \in \mathbb{N}$		n_2		$\ldots$
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Continue playing more rounds

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Determinacy of this game implies determinacy of the usual length ω game with payoff pA.

Use the same winning strategy for the first ω moves, to get a witness for $(n_0, n_1, \dots) \in pA$, play the second round of the longer game acc. to the strategy

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In fact, determinacy of these games is equivalent to projective determinacy.

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\end{array}
\quad \cdots
$$

resulting in a sequence
$(n_0, n_1, \ldots) \in \omega^{\omega \cdot \alpha}$

Games of countable length $\alpha > \omega$ and games on reals

Games	**Det $\Rightarrow$ Mice**	**Mice $\Rightarrow$ Det**
Analytic, length ω on $\mathbb{N}$	Martin, 1970	Harrington, 1978

$\rightarrow$ measurable cardinals

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$\rightarrow$ finitely many Woodin cardinals with a measurable ontop

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σ-projective, length ω on $\mathbb{N}$	Aguilera	Aguilera-M-Schlicht, APAL 2021, Aguilera

$\rightarrow$ "local" Woodin cardinals

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$\rightarrow$ $\omega+n$ many Woodin cardinals (for all $n<\omega$)

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Analytic, length ω^α on $\mathbb{N}$	Trang, 2013, building on Woodin	Neeman, 2004

$\rightarrow$ ω^α many Woodin cardinals
(for $\alpha < \omega_1$ limit)

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$\to \omega^\alpha + n$ many Woodin cardinals

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Projective, length ω^α on $\mathbb{N}$	M, 2020	Neeman, 2004
Projective, length ω on $\mathbb{R}$	Aguilera-M, NDJFL 2020	easy from Martin-Steel, 1989

Long games

Step 1: Fix a finite number n
and play games of length $\omega \cdot n$.

Step 2: Fix a countable ordinal α
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Step 3: Play games that end at a
countable stage but which one will
only be decided during the game.

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Theorem (Neeman, 2004)

Suppose there is an iterable proper class model M, with a Woodin cardinal δ and a cardinal $\kappa < \delta$ that is $(\delta+1)$-strong in M, such that $V^M_{\delta+1}$ is countable in V. Then the game $G_{\mathrm{cont}}(\nu, A)$ is determined for every ν in the class $\mathbf{\Sigma}^0_2$ and every A that is $<\omega^2 - \mathbf{\Pi}^1_1$ in the codes.

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Is this optimal?

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As an example, we sketch the proof of the following result.

Theorem (Aguilera-M, 2020)

Suppose every game of length $\omega^2 + \omega$ with a $\mathbf{\Pi}^1_1$ payoff set is determined. Then there is a model with $\omega + 1$ Woodin cardinals.

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Part II

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Play a game to get some $A \in \mathcal{P}_{\omega_1}(\mathbb{R})$ s.t. $M_1^\#(A)$ exists, $M_1^\#(A) \cap \mathbb{R} = A$, and

$M_1^\#(A) \models ZF + DC + AD + MC$.

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II		$b \in \mathbb{R}$

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The game:

I	$\overset{\in\mathbb{R}}{x_0}$ $\overset{\in\mathbb{R}}{a}$		$x_1 \in \mathbb{R}$	x_3	$\dots$
II		$b \in \mathbb{R}$	x_2	x_4	$\dots$

How could one prove such a result?

As an example, we sketch the proof of the following result.

Theorem (Aguilera-M, 2020)

Suppose every game of length $\omega^2 + \omega$ with a $\mathbf{\Pi}^1_1$ payoff set is determined. Then there is a model with $\omega + 1$ Woodin cardinals.

$\boxed{\text{Part I}}$

Play a game to get some $A \in \mathcal{P}_{\omega_1}(\mathbb{R})$ s.t. $M_1^\#(A)$ exists, $M_1^\#(A) \cap \mathbb{R} = A$, and

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I	$\overset{\in\mathbb{R}}{x_0}$	$\overset{\in\mathbb{R}}{a}$	$\overset{\in\{0,1\}}{v_0}, \overset{\in\mathbb{R}}{x_1}$		$\overset{\in\{0,1\}}{v_1}, x_3$	...
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I	$x_0 \in \mathbb{R}$, $a \in \mathbb{R}$		$v_0 \in \{0,1\}$, $x_1 \in \mathbb{R}$	$v_1 \in \{0,1\}$, x_3	...
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$$\begin{array}{c|cccccc} I & \overset{\in\mathbb{R}}{x_0} & \overset{\in\mathbb{R}}{a} & \overset{\in\{0,1\}}{v_0}, \overset{\in\mathbb{R}}{x_1} & & \overset{\in\{0,1\}}{v_1}, x_3 & \dots \\ \hline II & & b_{\in\mathbb{R}} & & x_2 & x_4 & \dots \end{array}$$

I wins iff the v_i's code a theory T s.t. models $\mathcal{N}$ of T are

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- there is a non-determined set in $\mathcal{N}$ definable from x_0

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The game:

I	$x_0 \in \mathbb{R}$	$a \in \mathbb{R}$	$v_0 \in \{0,1\},\ x_1 \in \mathbb{R}$	$v_1 \in \{0,1\},\ x_3$	$\dots$
II		$b \in \mathbb{R}$	x_2	x_4	$\dots$

I wins iff the v_i's code a theory T s.t. models N of T are

- $\{x_i \mid i<\omega\}$-mice
- there is a non-determined set in N definable from x_0 and if z is the least such, then $a \oplus b \in z$.

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$\boxed{\text{Part II}}$

Use the model $M_1^\#(A)$ from Part I and translate it into a model with $\omega+1$ Woodin cardinals.

A new game

"Future theorem" (Gappo-M, 2024)

The following are equivalent (over ZFC):

- There is a canonical inner model with a top measure and a limit of Woodin cardinals λ such that the order type of Woodin cardinals below λ is λ.
- The games $G(C)$ are determined for any $C \subset {}^{<\omega_1}\mathbb{R}$ that is $<\omega^2 - \Pi^1_1$ in the codes.

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Idea: The players determine the length of the game during the play.

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$$\begin{array}{c|l} \mathrm{I} & x_0 \in \mathbb{R} \\ \hline \mathrm{II} & \end{array}$$

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$\alpha_0 = \omega$ moves

let $\alpha_0 = \omega$.

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$$\begin{array}{c|ccccc} \mathrm{I} & x_0^{\in \mathbb{R}} & & x_2^{\in \mathrm{WO}} & & \dots \\ \hline \mathrm{II} & & x_1 & & x_3 & \dots \end{array}$$

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$\alpha_0 = \omega$ moves

Let $\alpha_0 = \omega$.
Let $\alpha_1 = \sup\{\|x_\beta\|_{WO} : \beta < \alpha_0 \wedge x_\beta \in WO\}$
($\alpha_1 > \alpha_0$, o/w stop).

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I	$x_0 \in \mathbb{R}$		$x_2 \in WO$	...	...	...
II		x_1	x_3 ...		...	

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$\vdots$

Stop the game at $\bigcup_{i\in\omega}\alpha_i$,

I wins iff $(x_\beta \mid \beta < \bigcup_{i\in\omega}\alpha_i) \in C$.

Long games

Step 1: Fix a finite number n
and play games of length $\omega \cdot n$.

Step 2: Fix a countable ordinal α
and play games of length $\omega \cdot \alpha$.

Step 3: Play games that end at a
countable stage but which one will
only be decided during the game.

Step 4: Play uncountable games.

Uncountable games

Be careful: The determinacy of arbitrary uncountable games is inconsistent with ZF (Mycielski, 1964).

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Consider games that are constructibly uncountable in the play.

Let $A \subseteq ({}^{\omega}\omega)^{<\omega_1}$.

I	$y_0(0)$		$y_0(2)$		$\dots$	$y_\alpha(0)$		$y_\alpha(2)$	$\dots$
II		$y_0(1)$		$y_0(3)$	$\dots$		$y_\alpha(1)$	$y_\alpha(3)$	$\dots$

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$$\begin{array}{c|cccccccc}
 & \multicolumn{3}{c}{\overbrace{\hphantom{y_0(0)\quad y_0(1)\quad y_0(2)\quad y_0(3)\quad\dots}}^{y_0}} & \dots & \multicolumn{3}{c}{\overbrace{\hphantom{y_\alpha(0)\quad y_\alpha(1)\quad y_\alpha(2)\quad y_\alpha(3)\quad\dots}}^{y_\alpha}} \\
\mathrm{I} & y_0(0) & & y_0(2) & \dots & y_\alpha(0) & & y_\alpha(2) & \dots \\
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$$\begin{array}{c|cccccccccc} & \multicolumn{4}{c}{\overbrace{\hphantom{y_0(0)\quad y_0(1)\quad y_0(2)\quad y_0(3)\quad \dots}}^{y_0}} & \dots & \multicolumn{4}{c}{\overbrace{\hphantom{y_\alpha(0)\quad y_\alpha(1)\quad y_\alpha(2)\quad y_\alpha(3)\quad \dots}}^{y_\alpha}} \\ \mathrm{I} & y_0(0) & & y_0(2) & & \dots & y_\alpha(0) & & y_\alpha(2) & & \dots \\ \hline \mathrm{II} & & y_0(1) & & y_0(3) & \dots & & y_\alpha(1) & & y_\alpha(3) & \dots \end{array}$$

The game ends when its length reaches the first $\gamma < \omega_1$ which is uncountable in $L[y_\alpha \mid \alpha < \gamma]$.

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The game ends when its length reaches the first $\gamma<\omega_1$ which is uncountable in $L[y_\alpha \mid \alpha<\gamma]$. I wins iff $(y_\alpha \mid \alpha<\gamma) \in A$.

Theorem (Neeman, 2004)

Suppose there is an iterable proper class model with a Woodin cardinal that is a limit of Woodin cardinals and countable in V. Then all games ending at ω_1 in L of the play with payoff sets that are $\Game(<\omega^2-\Pi^1_1)$ in the codes are determined.

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$\underbrace{\quad\quad\quad}_{y_0} \quad \dots \quad \underbrace{\quad\quad\quad}_{y_\alpha}$

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$\gamma = \omega_1^{L[y_\alpha \mid \alpha < \gamma]}$

Uncountable games

Theorem (Woodin)

The following theories are equiconsistent:

- ZFC + *all ordinal definable games of length* ω_1 *on natural numbers with real parameters are determined.*
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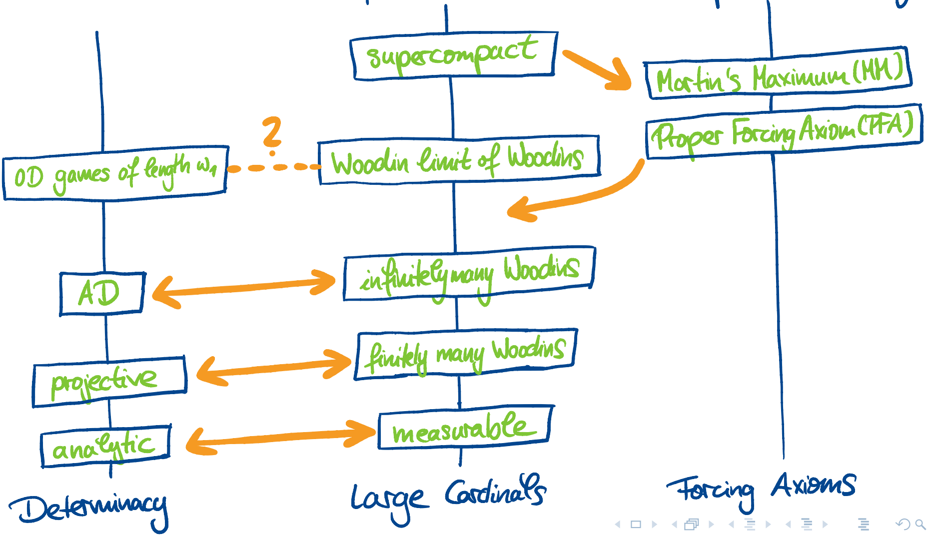
Is Neeman's result optimal,
i.e., are these theories equiconsistent
with a Woodin limit of Woodins?

How far are these axioms from ZFC? "Steel's Program"

Consider hierarchies of these axioms and compare their strength.



Two scenarios

What axiom(s) could fill the gap in the determinacy hierarchy?

Long games

Strong models

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